On Finding Integer Solutions to Binary Cubic Equation

$$3x^2 - xy = y^3$$

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ABSTRACT

The non-homogeneous polynomial equation of degree three with two unknowns given by $3x^2 - xy = y^3$ is studied to determine its distinct integer solutions . Some connections between the solutions are presented. Second order Ramanujan numbers are obtained through integer solutions of the given binary cubic equation .

KEYWORDS: Binary cubic equation ,Non-homogeneous cubic equation ,Integer solutions , Second order Ramanujan numbers

NOTATIONS

$$t_{3,n} = \frac{n(n+1)}{2}$$

$$P_n^5 = \frac{n^2(n+1)}{2}$$

$$P_n^3 = \frac{n(n+1)(n+2)}{6}$$

$$S_n = 6n(n+1)+1$$

$$Th_n = 3*2^n - 1$$

$$J_n = \frac{2^n - (-1)^n}{3}$$

$$j_n = 2^n + (-1)^n$$

INTRODUCTION

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics.The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of cubic Diophantine equations with multi variables [1-18]. It seems that much work has not been done regarding polynomial Diophantine equations of degree three with two unknowns. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree three with two unknowns given by $3x^2 - xy = y^3$. A few relations between the solutions are presented. A procedure for obtaining second order Ramanujan numbers through integer solutions of the given binary cubic equation is illustrated.

METHOD OF ANALYSIS

The non-homogeneous cubic equation with two unknowns under consideration is

$$3x^2 - xy = y^3 \tag{1}$$

Treating (1) as a quadratic in x and solving for the same ,we have

$$x = \frac{y[1 \pm \sqrt{12y+1}]}{6} \tag{2}$$

Consider the positive sign before the square-root in (2). After some calculations, it is seen that the square-root in (2) is removed when

$$y = y(s) = (3s-1)s \quad (3)$$

and correspondingly we have

$$x = x(s) = (3s-1)s^2 \quad (4)$$

Observe that (3) and (4) satisfy (1). A few numerical solutions to (1) are presented in Table-1 below:

Table-1: Numerical solutions

s	$y = y(s)$	$x = x(s)$
1	2	2
2	10	20
3	24	72
4	44	176
5	70	350
6	102	612
7	140	980

Relations observed

1. $y(s+2) - 2y(s+1) + y(s) \equiv 6$
2. $y(s+1) - y(s) - 1$ is a perfect square when $s = a(6a \pm 2)$
3. $y(s+1) - y(s) - 1 = S_a$ when $s = a(a+1)$
4. $x(s+2) - 2x(s+1) + x(s) = 18s + 16$
5. $x(s+1) - x(s) - s - 1$ is a perfect square
6. $x(s) + y(s) = 6P_s^5 - 2t_{3,s}$
7. $x(s+1) - y(s+1) = 6P_s^3 + 4P_s^5$
8. $x(s+1) - y(s+1) = 18P_s^3 - 4t_{3,s}$
9. $x(2^n) = (3J_{2n} + 1)\, Th_n$
10. $x(2^n) = (j_{2n} - 1) Th_n$
11. $x(s+2) - x(s+1) - s - 2$ is a perfect square
12. $x(s+2) - 2x(s+1) + x(s) - 1$ is expressed as difference of two squares
13. From the integer solutions one may obtain second order Ramanujan numbers as shown below:

Illustration

$$x(3)=72=1\times 72=2\times 36=3\times 24=4\times 18=6\times 12=8\times 9$$
$$= \quad F_1 \quad F_2 \quad F_3 \quad F_4 \quad F_5 \quad F_6$$

$$F_1=F_2 \Rightarrow (72+1)^2+(36-2)^2=(72-1)^2+(36+2)^2$$
$$=73^2+34^2=71^2+38^2=6485$$

$$F_1=F_3 \Rightarrow (72+1)^2+(24-3)^2=(72-1)^2+(24+3)^2$$
$$=73^2+21^2=71^2+27^2=5770$$

$$F_1=F_4 \Rightarrow (72+1)^2+(18-4)^2=(72-1)^2+(18+4)^2$$
$$=73^2+14^2=71^2+22^2=5525$$

$$F_1=F_5 \Rightarrow (72+1)^2+(12-6)^2=(72-1)^2+(12+6)^2$$
$$=73^2+6^2=71^2+18^2=5365$$

$$F_1=F_6 \Rightarrow (72+1)^2+(9-8)^2=(72-1)^2+(9+8)^2$$
$$=73^2+1^2=71^2+17^2=5330$$

$$F_2=F_3 \Rightarrow (36+2)^2+(24-3)^2=(36-2)^2+(24+3)^2$$
$$=38^2+21^2=34^2+27^2=1885$$

$$F_2=F_4 \Rightarrow (36+2)^2+(18-4)^2=(36-2)^2+(18+4)^2$$
$$=38^2+14^2=34^2+22^2=1640$$

$$F_2=F_5 \Rightarrow (36+2)^2+(12-6)^2=(36-2)^2+(12+6)^2$$
$$=38^2+6^2=34^2+18^2=1480$$

$$F_2=F_6 \Rightarrow (36+2)^2+(9-8)^2=(36-2)^2+(9+8)^2$$
$$=38^2+1^2=34^2+17^2=1445$$

$$F_3=F_4 \Rightarrow (24+3)^2+(18-4)^2=(24-3)^2+(18+4)^2$$
$$=27^2+14^2=21^2+22^2=925$$

$$F_3=F_5 \Rightarrow (24+3)^2+(12-6)^2=(24-3)^2+(12+6)^2$$
$$=27^2+6^2=21^2+18^2=765$$

$$F_3=F_6 \Rightarrow (24+3)^2+(9-8)^2=(24-3)^2+(9+8)^2$$
$$=27^2+1^2=21^2+17^2=730$$

$$F_4=F_5 \Rightarrow (18+4)^2+(12-6)^2=(18-4)^2+(12+6)^2$$
$$=22^2+6^2=14^2+18^2=520$$

$$F_4=F_6 \Rightarrow (18+4)^2+(9-8)^2=(18-4)^2+(9+8)^2$$
$$=22^2+1^2=14^2+17^2=485$$

$$F_5=F_6 \Rightarrow (12+6)^2+(9-8)^2=(12-6)^2+(9+8)^2$$
$$=18^2+1^2=6^2+17^2=325$$

Thus , 6485,5770,5525,5365,5330,1885,1640,1480,1445,925,765, 730, 520,485,325 represent second order Ramanujan numbers .

A similar observation may be performed by considering the other solutions.

Remark 1

Taking the negative sign before the square-root in (2) and performing some algebra, the corresponding integer solutions to (1) are given by

$$y = y(s) = (3s+1)s, x = x(s) = -(3s+1)s^2$$

Note 1

To remove the square-root in (2) , let

$$\alpha^2 = 12y + 1 \tag{5}$$

which, after some algebra , is satisfied by

$$y_0 = s(3s-1) \ , \alpha_0 = 6s - 1 \tag{6}$$

Assume the second solution to (5) as

$$\alpha_1 = h - \alpha_0 , y_1 = h + y_0 \tag{7}$$

where h is an unknown to be determined. Substituting (7) in (5) and

simplifying, we have

$$h = 2\alpha_0 + 12$$

and in view of (7) ,it is seen that

$$\alpha_1 = \alpha_0 + 12, y_1 = y_0 + 2\alpha_0 + 12$$

The repetition of the above process leads to the general solution to (5) as

$$\begin{aligned} \alpha_n &= \alpha_0 + 12n = 6s - 1 + 12n, \\ y_n &= y_n(s) = y_0 + 2n\alpha_0 + 12n^2 = 3(s+2n)^2 - (s+2n) \end{aligned} \tag{8}$$

Taking the positive sign before the square-root in (2) ,we get

$$\begin{aligned} x_n = x_n(s) &= \frac{y_n(s)\,[1+\alpha_n]}{6} \\ &= (2n+s)\,y_n(s) \end{aligned} \tag{9}$$

Thus, the integer solutions to (1) are represented by (8) and (9).

Relations observed

(i) $y_{n+2}(s)-2y_{n+1}(s)+y_n(s)=24, n=0,1,2,...$

(ii) $16[y_n(s)]^3+16[y_n(s)]^2=96\,t_{3,s+2n}[y_n(s)]^2+[x_n(s)-3y_n(s)]^2$

(iii) $\frac{x_n(s)}{y_n(s)}=2\,P_n^5$ when $s=n^3+(n-1)^2-1$

(iv) $y_n(s+2)-2y_n(s+1)+y_n(s)=6, n=0,1,2,...$

(v) $y_n(s)[y_n(s+1)-y_n(s)-2]=6x_n(s)$

(vi) $y_n(s)[y_n(s+2)-y_n(s)]=12x_n(s)+10\,y_n(s)$

(vii) $y_n(s)[y_{n+2}(s)-y_{n+1}(s)]=12x_n(s)+34\,y_n(s)$

(viii) $y_n(s)[y_n(s+2)-y_n(s+1)]=6x_n(s)+8\,y_n(s)$

(ix) $\frac{x_n(s)}{y_n(s)}$ is a perfect square when $s=n^2+2(k-i)n+k^2$

(x) $y_{n+1}(s)=y_n(s+2), x_{n+1}(s)=x_n(s+2)$

(xi) $\frac{x_n(s)}{y_n(s)}$ is a Star number of rank n (S_n) when $s=6n^2+4n+1$

(xii) $\frac{x_n(s)}{y_n(s)}$ is a Thabit ibn Kurrah number of rank n (Th_n) when $s=3*2^n-2n-1$

(xiii) $\frac{x_n(s)}{y_n(s)}=2t_{3,n}$ when $s=n^2-n$

Remark 2

Apart from (6), (5) is satisfied by

$$y_0=s(3s+1)\ ,\alpha_0=6s+1$$

For this choice, we have

$$\alpha_n=\alpha_0+12n=6s+1+12n,$$
$$y_n=y_0+2n\alpha_0+12n^2=3(s+2n)^2+(s+2n) \quad (10)$$

Taking the negative sign before the square-root in (2) ,we get

$$x_n = \frac{y_n[1-\alpha_n]}{6}$$
$$= -(2n+s)\,y_n \quad (11)$$

Thus, the integer solutions to (1) are represented by (10)&(11).

Conclusion

The polynomial equation of degree three with two unknowns has been studied to obtain non-zero integer solutions .The process of eliminating the square-root will be beneficial for the researchers. As cubic equations are plenty, one may attempt to determine the solutions in integers for other choices of cubic diophantine equations.

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