ON SOLVING NON-HOMOGENEOUS TERNARY QUINTIC DIOPHANTINE EQUATION

$$w^2 + 2z^2 - 2wx - 4zx = x^5 - 3x^2$$

J.Shanthi and M.A.Gopalan

Department of Mathematics, Shrimati Indira Gandhi College,Affiliated to Bharathidasan University, Trichy-620 002,
Tamil Nadu, India.
Dr. S. DEVIBALA,

Associate Professor of Mathematics,
Sri Meenakshi Govt Arts College for women(A),
Affiliated to Madurai Kamaraj University,
Madurai, Tamil Nadu, India.

ABSTRACT:

The non-homogeneous ternary fifth degree Diophantine equation given by $w^2 + 2z^2 - 2wx - 4zx = x^5 - 3x^2$ is analyzed for its patterns of non-zero distinct integral solutions..

KEYWORDS: Ternary quintic equation ,Non- Homogeneous quintic equation , Integral solutions

INTRODUCTION:

The Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-10] for quintic equations with two ,three and five unknowns. This communication concerns with yet another interesting equation $w^2 + 2z^2 - 2wx - 4zx = x^5 - 3x^2$ representing non-homogeneous quintic with three unknowns for determining its infinitely many non-zero integral points.

METHOD OF ANALYSIS:

The given non-homogeneous ternary quintic Diophantine equation is

$$w^2 + 2z^2 - 2wx - 4zx = x^5 - 3x^2 \tag{1}$$

On completing the squares,(1) is written as

$$P^2 + 2Q^2 = x^5 \tag{2}$$

where

$$P = w - x, Q = z - x \tag{3}$$

We illustrate below the process of obtaining different sets of integer solutions to (1):

Set 1:

After some algebra , it is observed that (2) is satisfied by

$$P = m(m^2 + 2n^2)^2, Q = n(m^2 + 2n^2)^2 \tag{4}$$

and

$$x = m^2 + 2n^2 \tag{5}$$

From (4) and (3) ,we have

$$w = [m(m^2 + 2n^2) + 1](m^2 + 2n^2), z = [n(m^2 + 2n^2) + 1](m^2 + 2n^2) \tag{6}$$

Thus,(5) and (6) represent the integer solutions to (1).

Set 2:

Assuming (5) in (2) and employing the method of factorization ,one obtains

$$P + i\sqrt{2}Q = (m + i\sqrt{2}n)^5 \tag{7}$$

On equating the real and imaginary parts ,we have

$$P = f(m,n), Q = g(m,n) \tag{8}$$

where

$$f(m,n) = m^5 - 20m^3n^2 + 20mn^4,$$
$$g(m,n) = 5m^4n - 20m^2n^3 + 4n^5$$

Using (8) in (3), note that

$$z = m^2 + 2n^2 + g(m,n), w = m^2 + 2n^2 + f(m,n) \tag{9}$$

Thus,(5) and (9) represent the integer solutions to (1).

Set 3:

Write (2) as

$$P^2 + 2Q^2 = x^5 * 1 \tag{10}$$

Consider 1 as

$$1 = \frac{(F(r,s) + i\sqrt{2}\, G(r,s))\ (F(r,s) - i\sqrt{2}\, G(r,s))}{[H(r,s)]^2} \tag{11}$$

where

$$F(r,s) = 2r^2 - s^2, G(r,s) = 2rs, H(r,s) = 2r^2 + s^2$$

Using (5) and (11) in (10) and employing the method of factorization, one has

$$P + i\sqrt{2}Q = \frac{(F(r,s) + i\sqrt{2}\,G(r,s))\,(m + i\sqrt{2}n)^5}{H(r,s)} \quad (12)$$

Equating the real and imaginary parts in (12) and replacing m by $H(r,s)M$ and n by $H(r,s)N$,we get

$$\begin{aligned} P &= [H(r,s)]^4 [f(M,N) F(r,s) - 2g(M,N)\, G(r,s)] \\ Q &= [H(r,s)]^4 [f(M,N) G(r,s) + g(M,N)\, F(r,s)] \end{aligned} \quad (13)$$

Also ,from (5) ,we have

$$x = [H(r,s)]^2 (M^2 + 2N^2) \quad (14)$$

Substituting (13) and (14) in (3) ,one obtains the corresponding values of z, w

satisfying (1) .

Set 4 :

The option

$$Q = kx^2 \quad (15)$$

in (2) leads to

$$P^2 = x^4 (x - 2k^2)$$

which is satisfied by

$$x = (s^2 + 2)k^2 \quad (16)$$

and

$$P = s\,(s^2 + 2)^2 k^5 \quad (17)$$

Using (16) in (15) , we have

$$Q = (s^2 + 2)^2 k^5 \quad (18)$$

Substituting (17) and (18) in (3) , it is seen that

$$\begin{aligned} w &= (s^2 + 2)k^2 [s(s^2 + 2)k^3 + 1] \\ z &= (s^2 + 2)k^2 [(s^2 + 2)k^3 + 1] \end{aligned} \quad (19)$$

Thus , (16) and (19) satisfy (1).

Set 5 :

The option

$$P = x^2 \qquad (20)$$

in (2) leads to

$$2Q^2 = x^4 (x-1)$$

which is satisfied by

$$x = (2k^2+1) \qquad (21)$$

and

$$Q = k\,(2k^2+1)^2 \qquad (22)$$

Using (21) in (20) , we have

$$P = (2k^2+1)^2 \qquad (23)$$

Substituting (21) ,(22) and (23) in (3) , it is seen that

$$\begin{aligned} w &= (2k^2+1)(2k^2+2) \\ z &= (2k^2+1)\,[k(2k^2+1)+1] \end{aligned} \qquad (24)$$

Thus , (21) and (24) satisfy (1).

Set 6 :

The choice

$$P = kQ \qquad (25)$$

in (2) gives

$$(k^2+2)\,Q^2 = x^5$$

which is satisfied by

$$Q = (k^2+2)^2\,\alpha^{5s} \qquad (26)$$

and

$$x = (k^2+2)\,\alpha^{2s} \qquad (27)$$

From (25) ,it is seen that

$$P = k(k^2+2)^2\,\alpha^{5s} \qquad (28)$$

Using (26) ,(27) and (28) in (3), we have

$$\begin{aligned} w &= k(k^2+2)^2\,\alpha^{5s} + (k^2+2)\,\alpha^{2s} \\ z &= (k^2+2)^2\,\alpha^{5s} + (k^2+2)\,\alpha^{2s} \end{aligned} \qquad (29)$$

Thus , (1) is satisfied by (27) and (29).

Set 7 :

The choice

$$Q = k P$$

(30) in (2) gives

$$(2k^2 + 1) P^2 = x^5$$

which is satisfied by

$$P = (2k^2 + 1)^2 \alpha^{5s} \quad (31)$$

and

$$x = (2k^2 + 1) \alpha^{2s} \quad (32)$$

From (30) ,it is seen that

$$Q = k(2k^2 + 1)^2 \alpha^{5s} \quad (33)$$

Using (31) ,(32) and (33) in (3), we have

$$\begin{aligned} w &= (2k^2 + 1)^2 \alpha^{5s} + (2k^2 + 1) \alpha^{2s} \\ z &= k(2k^2 + 1)^2 \alpha^{5s} + (2k^2 + 1) \alpha^{2s} \end{aligned} \quad (34)$$

Thus , (1) is satisfied by (32) and (34).

CONCLUSION:

In this paper, we have made an attempt to obtain all integer solutions to (1). To conclude, one may search for integer solutions to other choices of homogeneous or non-homogeneous quintic equations with multiple variables.

REFERENCES

[1].L.E. Dickson, History of Theory of Numbers, vol 2, Chelsea publishing company, New York, (1952).

[2].L.J. Mordell, Diophantine Equations, Academic press, London, (1969).

[3].R.D. Carmichael, The theory of numbers and Diophantine analysis, New York, Dover, (1959).

[4] S.G. Telang. Number theory, Tata Mc Graw Hill publishing company, New Delhi . 1996

[5] J.Shanthi , M.A.Gopalan , On finding Integer Solutions to Binary Quintic Equation $x^2 - xy^2 = y^5$, International Research Journal of Education and Technology, 6(10), 226-231, 2024.

[6] J.Shanthi , M.A.Gopalan , On finding Integer Solutions to Binary Quintic Equation $x^2 - xy^2 = y^4 + y^5$, International Journal of Advanced Research in Science ,Engineering and Technology, 11(10), 22369-22372, 2024.

[7] J.Shanthi ,M.A.Gopalan , On the Ternary Non-homogeneous Quintic Equation $x^2 + 5y^2 = 2z^5$, International Journal of Research Publication and Reviews,4(8),329-332,2023.

[8] M. A. Gopalan, S. Vidhyalakshmi, J. Shanthi, " The non-homogeneous quintic

equation with five unknowns $x^4 - y^4 + 2k\left(x^2 + y^2\right)\left(x - y + k\right) = \left(a^2 + b^2\right)\left(z^2 - w^2\right)p^{3n}$ ", open journal of applied and theoretical mathematics, vol-2, no 3, 08-13, sep 2016.

[9] J.Shanthi ,M.A.Gopalan , Delineation of Integer Solutions to Non-Homogeneous Quinary Quintic Diophantine Equation $(x^3 - y^3) - (x^2 + y^2) + (z^3 - w^3) = 2 + 87T^5$, International Journal of Research Publication and Reviews, 4(9), 1454-1457 , 2023.

[10] J.Shanthi ,M.A.Gopalan , On the Non-homogeneous Quinary Quintic Equation $x^4 + y^4 - (x + y)w^3 = 14z^2T^3$, International Research Journal of Education and Technology , 05 (08) ,238-245 , 2023.