

**Exploring Solutions to the Negative Pell Equation  $y^2 = 23x^2 - 11$**

**Dr. Elena Rodriguez**

**Assistant Professor, Department of Mathematics, University of Valencia, Valencia, Spain**

**ABSTRACT**

The binary quadratic Diophantine equation represented by the negative pellian  $y^2 = 23x^2 - 11$  is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

*Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.*

**I. INTRODUCTION**

The binary quadratic equations of the form  $y^2 = Dx^2 + 1$  where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-10]. In this communication, yet another interesting equation given by  $y^2 = 23x^2 - 11$  is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

**II. METHOD OF ANALYSIS**

The negative Pell equation representing hyperbola under consideration is,

$$y^2 = 23x^2 - 11 \tag{1}$$

The smallest negative integer solutions of (1) are,

$$x_0 = 2, y_0 = 9$$

The pellian equation is

$$y^2 = 23x^2 + 1 \tag{2}$$

The initial solution of pellian equation is

$$\tilde{x}_0 = 5, \tilde{y}_0 = 24$$

The general solution  $(x_n, y_n)$  of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{23}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (24 + 5\sqrt{23})^{n+1} - (24 - 5\sqrt{23})^{n+1}$$

$$g_n = (24 + 5\sqrt{23})^{n+1} + (24 - 5\sqrt{23})^{n+1} \quad n = 0, 1, 2, \dots$$

Applying Brahmagupta lemma between  $(x_0, y_0)$  and  $(\tilde{x}_n, \tilde{y}_n)$  the other integer solution of (1) are given by,

$$x_{n+1} = f_n + \frac{9}{2\sqrt{23}} g_n$$

$$y_{n+1} = \frac{9}{2}f_n + \frac{23}{\sqrt{23}}g_n$$

The recurrence relation satisfied by the solution  $x$  and  $y$  are given by,

$$x_{n+3} - 48x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 48y_{n+2} + y_{n+1} = 0 \quad n=0,1,2,3,\dots$$

Some numerical examples of  $x$  and  $y$  satisfying (1) are given in the Table 1 below,

**Table 1: Examples**

$n$	$x_n$	$y_n$
0	2	9
1	93	446
2	4462	21399
3	214083	1026706

From the above table, we observe some interesting relations among the solutions which are presented below.

$x_n$  values are even and odd alternatively.

$y_n$  values are odd and even alternatively.

Each of the following expression is a nasty number:

$$\frac{6}{55} [110 + 892x_{2n+2} - 18x_{2n+4}]$$

$$\frac{6}{440} [880 + 7133x_{2n+2} - 3x_{2n+4}]$$

$$\frac{6}{44} [88 + 713x_{2n+2} - 3y_{2n+3}]$$

$$\frac{6}{12661} [25322 + 205252x_{2n+2} - 18y_{2n+4}]$$

$$\frac{1}{11} [132 + 23x_{2n+4} - 223y_{2n+3}]$$

$$\frac{6}{12661} [25322 + 92x_{2n+4} - 42798y_{2n+3}]$$

$$\frac{6}{1265} [2530 + 92y_{2n+3} - 4278y_{2n+2}]$$

$$\frac{1}{220} [2640 + 2y_{2n+4} - 4462y_{2n+3}]$$

$$\frac{6}{55} [110 + 42798x_{2n+3} - 892x_{2n+4}]$$

$$\frac{6}{11} [22 + 4278x_{2n+3} - 892y_{2n+3}]$$

$$\frac{1}{11} [132 + 223y_{2n+4} - 51313x_{2n+3}]$$

$$\frac{6}{44} [88 + 713x_{2n+4} - 7133y_{2n+3}]$$

$$\frac{6}{11} [22 + 205252x_{2n+4} - 42798y_{2n+4}]$$

$$\frac{6}{1265} [2530 + 4278y_{2n+4} - 205252y_{2n+3}]$$

Each of the following expressions is a cubical integer.

$$\frac{1}{55} [892x_{3n+3} - 18x_{3n+5} + 2676y_{n+1} - 54x_{n+1}]$$

$$\frac{1}{440} [7133x_{3n+3} - 3x_{3n+5} + 21399x_{n+1} - 9x_{n+3}]$$

$$\frac{1}{12661} [205252x_{3n+3} - 18y_{3n+5} + 615756x_{n+1} - 54y_{n+3}]$$

$$\frac{1}{44} [713x_{3n+3} - 3y_{3n+4} + 2139x_{n+1} - 9y_{n+2}]$$

$$\frac{1}{66} [23x_{3n+5} - 223y_{3n+4} + 69x_{n+2} - 669y_{n+1}]$$

$$\frac{1}{12661} [92x_{3n+5} - 42798y_{3n+4} + 276x_{n+3} - 128394y_{n+1}]$$

$$\frac{1}{1265} [92y_{3n+4} - 4278y_{3n+3} + 276y_{n+2} - 12834y_{n+1}]$$

$$\frac{1}{660} [y_{3n+5} - 2231y_{3n+4} + 3y_{n+3} - 6693y_{n+1}]$$

$$\frac{1}{55} [42798x_{3n+4} - 892x_{3n+5} + 128394x_{n+2} - 2676x_{n+3}]$$

$$\frac{1}{11} [4278x_{3n+4} - 892y_{3n+4} + 12834x_{n+2} - 2676y_{n+2}]$$

$$\frac{1}{66} [223y_{3n+5} - 51313x_{3n+4} + 669y_{n+3} - 153939x_{n+2}]$$

$$\frac{1}{44} [713x_{3n+5} - 7133y_{3n+4} + 2139x_{n+3} - 21399y_{n+2}]$$

$$\frac{1}{11} [205252x_{3n+5} - 42798y_{3n+5} + 615756x_{n+3} - 128394y_{n+3}]$$

$$\frac{1}{1265} [4278y_{3n+5} - 205252y_{3n+4} + 12834y_{n+3} - 615756y_{n+2}]$$

Each of the following expressions is a biquadratic integer.

$$\frac{1}{55} [892x_{4n+4} - 18x_{4n+6} + 3568x_{2n+2} - 72x_{2n+4} + 330]$$

$$\frac{1}{440} [7133x_{4n+4} - 3x_{4n+6} + 28532x_{2n+2} - 12x_{2n+4} + 2640]$$

$$\frac{1}{44} [713x_{4n+4} - 3y_{4n+5} + 2852x_{2n+2} - 12y_{2n+3} + 264]$$

$$\frac{1}{12661} \left[ \begin{array}{l} 205252x_{4n+4} - 18y_{4n+6} + 821008x_{2n+2} - 72y_{2n+4} \\ + 75966 \end{array} \right]$$

$$\frac{1}{66} [23x_{4n+6} - 223y_{4n+5} + 92x_{2n+4} - 892y_{2n+3} + 396]$$

$$\frac{1}{12661} [92x_{4n+6} - 42798y_{4n+5} + 368x_{2n+4} - 171192y_{2n+3} + 75966]$$

$$\frac{1}{1265} [92y_{4n+5} - 4278y_{4n+4} + 36y_{2n+3} - 17112y_{2n+2} + 7590]$$

$$\frac{1}{660} [y_{4n+6} - 2231y_{4n+5} + 4y_{2n+4} - 8924y_{2n+3} + 2640]$$

$$\frac{1}{55} [42798x_{4n+5} - 892x_{4n+6} + 171198x_{2n+3} - 3568x_{2n+4} + 330]$$

$$\frac{1}{11} [12426y_{4n+5} - 1242x_{4n+6} + 49704y_{2n+3} - 4968x_{2n+4} + 936]$$

$$\frac{1}{66} [223y_{4n+6} - 51313x_{4n+5} + 892y_{2n+4} - 205252x_{2n+3} + 396]$$

$$\frac{1}{44} [713x_{4n+6} - 7133y_{4n+5} + 2852x_{2n+4} - 28532y_{2n+3} + 264]$$

$$\frac{1}{1265} [4278y_{4n+6} - 205252y_{4n+5} + 17112y_{2n+4} - 821008y_{2n+3}]$$

$$\frac{1}{11} \left[ \begin{array}{l} 205252x_{4n+6} - 42798y_{4n+6} + 821008x_{2n+4} - 171192y_{2n+4} \\ + 66 \end{array} \right]$$

Each of the following expression is a quintic integer:

$$\frac{1}{55} \left[ \begin{array}{l} 892x_{5n+5} - 18x_{5n+7} + 4460x_{3n+3} - 90x_{3n+5} \\ + 8920x_{n+1} - 180y_{n+1} \end{array} \right]$$

$$\frac{1}{440} \left[ \begin{array}{l} 7133x_{5n+5} - 3x_{5n+7} - 356655x_{3n+3} - 15x_{3n+5} \\ + 71330x_{n+1} - 30x_{n+3} \end{array} \right]$$

$$\frac{1}{44} \left[ \begin{array}{l} 713x_{5n+5} - 3y_{5n+6} + 3565x_{3n+3} - 15y_{3n+4} \\ + 7130x_{n+1} - 30y_{n+2} \end{array} \right]$$

$$\frac{1}{12661} \left[ \begin{array}{l} 205252x_{5n+5} - 18y_{5n+7} + 1026260x_{3n+3} - 90y_{3n+5} + \\ 2052520x_{n+1} - 180y_{n+3} \end{array} \right]$$

$$\frac{1}{66} \left[ \begin{array}{l} 23x_{5n+7} - 223y_{5n+6} + 115x_{3n+5} - 1115y_{3n+4} \\ + 230x_{n+2} - 2230y_{n+1} \end{array} \right]$$

$$\frac{1}{12661} \left[ \begin{array}{l} 92x_{5n+7} - 42798y_{5n+6} + 460x_{3n+5} - 213990y_{3n+4} \\ + 920x_{n+3} - 427980y_{n+1} \end{array} \right]$$

$$\frac{1}{1265} \left[ \begin{array}{l} 92y_{5n+6} - 4278y_{5n+5} + 460y_{3n+4} - 21390y_{3n+3} \\ + 920y_{n+2} - 42780y_{n+1} \end{array} \right]$$

$$\frac{1}{660} \left[ \begin{array}{l} y_{5n+7} - 2231y_{5n+6} + 5y_{3n+5} - 11155y_{3n+4} \\ + 20y_{n+3} - 22310y_{n+1} \end{array} \right]$$

$$\frac{1}{55} \left[ \begin{array}{l} 42798x_{5n+6} - 892x_{5n+6} + 213990x_{3n+4} - 4460x_{3n+5} \\ + 427980x_{n+2} - 8920x_{n+3} \end{array} \right]$$

$$\frac{1}{11} \left[ \begin{array}{l} 42798x_{5n+6} - 892x_{5n+7} + 21390x_{3n+4} - 4460y_{3n+4} \\ + 42780x_{n+2} - 8920y_{n+2} \end{array} \right]$$

$$\frac{1}{66} \left[ \begin{array}{l} 223y_{5n+7} - 51313x_{5n+6} + 1115y_{3n+5} - 256565x_{3n+4} \\ + 2230y_{n+3} - 513130x_{n+2} \end{array} \right]$$

$$\frac{1}{44} \left[ \begin{array}{l} 713x_{5n+7} - 7133y_{5n+6} + 3565x_{3n+5} - 35665y_{3n+4} \\ + 7130x_{n+3} - 71330y_{n+2} \end{array} \right]$$

$$\frac{1}{11} \left[ \begin{array}{l} 205252x_{5n+7} - 42798y_{5n+7} + 1026260x_{3n+5} - 213990y_{3n+5} \\ + 2052520x_{n+3} - 427980y_{n+3} \end{array} \right]$$

$$\frac{1}{1265} \left[ \begin{array}{l} 4278y_{5n+7} - 205252y_{5n+6} + 21390y_{3n+5} - 1026260y_{3n+4} \\ + 42780y_{n+3} - 2052520y_{n+2} \end{array} \right]$$

Relations among the solutions are given below:

$$x_{n+2} = 5y_{n+1} + 24x_{n+1}$$

$$x_{n+3} = 240y_{n+1} + 1151x_{n+1}$$

$$y_{n+2} = 24y_{n+1} + 115x_{n+1}$$

$$y_{n+3} = 1151y_{n+1} + 5520x_{n+1}$$

$$x_{n+3} = 48x_{n+2} - x_{n+1}$$

$$5y_{n+2} = 24x_{n+2} - x_{n+1}$$

$$5y_{n+3} = 1151x_{n+2} - 24x_{n+1}$$

$$10y_{n+2} = x_{n+3} - x_{n+1}$$

$$240y_{n+3} = 1151x_{n+3} - x_{n+1}$$

$$24y_{n+3} = 1151y_{n+2} + 115x_{n+1}$$

$$24x_{n+3} = 5y_{n+1} + 1151x_{n+2}$$

$$24y_{n+2} = y_{n+1} + 115x_{n+2}$$

$$24y_{n+3} = 24y_{n+1} + 5520x_{n+2}$$

$$1151y_{n+2} = 24y_{n+1} + 115x_{n+3}$$

$$1151y_{n+3} = y_{n+1} + 5520x_{n+3}$$

$$115y_{n+3} = 5520y_{n+2} - 115y_{n+1}$$

$$5y_{n+2} = x_{n+3} - 24x_{n+2}$$

$$5y_{n+3} = 24x_{n+3} - x_{n+2}$$

$$y_{n+3} = 24y_{n+2} + 115x_{n+2}$$

$$24y_{n+3} = y_{n+2} + 115x_{n+3}$$

**Remarkable Observation**

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

*Table 2: Hyperbola*

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 23X^2 = 484$	$\left( \begin{matrix} 4y_{n+1} - 18x_{n+1}, \\ 92x_{n+1} - 18y_{n+1} \end{matrix} \right)$
2	$Y^2 - 23X^2 = 12100$	$\left( \begin{matrix} 4x_{n+2} - 186x_{n+1}, \\ 892x_{n+1} - 18x_{n+2} \end{matrix} \right)$
3	$Y^2 - 23X^2 = 6969600$	$\left( \begin{matrix} 2x_{n+3} - 4462x_{n+1}, \\ 21399x_{n+1} - 9x_{n+3} \end{matrix} \right)$
4	$Y^2 - 23X^2 = 69696$	$\left( \begin{matrix} 2y_{n+2} - 446x_{n+1}, \\ 2139x_{n+1} - 9y_{n+2} \end{matrix} \right)$
5	$Y^2 - 23X^2 = 641203684$	$\left( \begin{matrix} 4y_{n+3} - 42798x_{n+1}, \\ 205252x_{n+1} - 18y_{n+3} \end{matrix} \right)$
6	$Y^2 - 23X^2 = 69696$	$\left( \begin{matrix} 93y_{n+1} - 9x_{n+2}, \\ 46x_{n+2} - 446y_{n+1} \end{matrix} \right)$
7	$Y^2 - 23X^2 = 641203684$	$\left( \begin{matrix} 8924y_{n+1} - 18x_{n+3}, \\ 92x_{n+3} - 42798y_{n+1} \end{matrix} \right)$
8	$Y^2 - 23X^2 = 3686918400$	$\left( \begin{matrix} 12y_{n+2} - 518y_{n+1}, \\ 2484y_{n+1} - 46y_{n+2} \end{matrix} \right)$
9	$Y^2 - 23X^2 = 5149497600$	$\left( \begin{matrix} 21399y_{n+1} - 9y_{n+3}, \\ 46y_{n+3} - 102626y_{n+1} \end{matrix} \right)$
10	$Y^2 - 23X^2 = 12100$	$\left( \begin{matrix} 186x_{n+3} - 8924x_{n+2}, \\ 42798x_{n+2} - 892x_{n+3} \end{matrix} \right)$

11	$Y^2 - 23X^2 = 484$	$\left( \begin{array}{l} 18y_{n+2} - 892x_{n+2}, \\ 4278x_{n+2} - 892y_{n+2} \end{array} \right)$
12	$Y^2 - 23X^2 = 69696$	$\left( \begin{array}{l} 21399x_{n+2} - 93y_{n+3}, \\ 446y_{n+3} - 102626x_{n+2} \end{array} \right)$
13	$Y^2 - 23X^2 = 69696$	$\left( \begin{array}{l} 4462y_{n+2} - 446x_{n+3}, \\ 2139x_{n+3} - 21399y_{n+2} \end{array} \right)$
14	$Y^2 - 23X^2 = 484$	$\left( \begin{array}{l} 8924y_{n+3} - 42798x_{n+3}, \\ 205252x_{n+3} - 42798y_{n+3} \end{array} \right)$
15	$Y^2 - 23X^2 = 6400900$	$\left( \begin{array}{l} 42798y_{n+2} - 892y_{n+3}, \\ 4278y_{n+3} - 205252y_{n+2} \end{array} \right)$

Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

**Table 3: Parabola**

S.NO	Parabola	(X,Y)
1	$11Y - 23X^2 = 484$	$\left( \begin{array}{l} 4y_{n+1} - 18x_{n+1}, \\ 92x_{2n+2} - 18y_{2n+2} + 22 \end{array} \right)$
2	$55Y - 23X^2 = 12100$	$\left( \begin{array}{l} 4x_{n+2} - 186x_{n+1}, \\ 892x_{2n+2} - 18x_{2n+3} + 110 \end{array} \right)$
3	$1320Y - 23X^2 = 6969600$	$\left( \begin{array}{l} 2x_{n+3} - 4462x_{n+1}, \\ 21399x_{2n+2} - 9x_{2n+4} + 2640 \end{array} \right)$
4	$132Y - 23X^2 = 69696$	$\left( \begin{array}{l} 2y_{n+2} - 446x_{n+1}, \\ 2139x_{2n+2} - 9y_{2n+3} + 264 \end{array} \right)$
5	$12661Y - 23X^2 = 641203684$	$\left( \begin{array}{l} 4y_{n+3} - 42798x_{n+1}, \\ 205252x_{2n+2} - 18y_{2n+4} + 25322 \end{array} \right)$
6	$132Y - 23X^2 = 69696$	$\left( \begin{array}{l} 93y_{n+1} - 9x_{n+2}, \\ 46x_{2n+3} - 446y_{2n+2} + 264 \end{array} \right)$

7	$12661Y - 23X^2 = 641203684$	$\left( \begin{array}{l} 8924y_{n+1} - 18x_{n+3}, \\ 92x_{2n+4} - 42798y_{2n+2} + 25322 \end{array} \right)$
8	$1265Y - 23X^2 = 3686918400$	$\left( \begin{array}{l} 12y_{n+2} - 518y_{n+1}, \\ 2484y_{2n+2} - 46y_{2n+3} + 2530 \end{array} \right)$
9	$30360Y - 23X^2 = 5149497600$	$\left( \begin{array}{l} 21399y_{n+1} - 9y_{n+3}, \\ 46y_{2n+4} - 102626y_{2n+2} + 60720 \end{array} \right)$
10	$55Y - 23X^2 = 12100$	$\left( \begin{array}{l} 186x_{n+3} - 8924x_{n+2}, \\ 42798x_{2n+3} - 892x_{2n+4} + 110 \end{array} \right)$
11	$11Y - 23X^2 = 484$	$\left( \begin{array}{l} 18y_{n+2} - 892x_{n+2}, \\ 4278x_{2n+3} - 892y_{2n+3} + 22 \end{array} \right)$
12	$132Y - 23X^2 = 69696$	$\left( \begin{array}{l} 21399x_{n+2} - 93y_{n+3}, \\ 446y_{2n+4} - 102626x_{2n+4} + 264 \end{array} \right)$
13	$132Y - 23X^2 = 69696$	$\left( \begin{array}{l} 4462y_{n+2} - 446x_{n+3}, \\ 2139x_{2n+4} - 21399y_{2n+3} + 264 \end{array} \right)$
14	$11Y - 23X^2 = 484$	$\left( \begin{array}{l} 8924y_{n+3} - 42798x_{n+3}, \\ 205252x_{2n+4} - 42798y_{2n+4} + 22 \end{array} \right)$
15	$1265Y - 23X^2 = 6400900$	$\left( \begin{array}{l} 42798y_{n+2} - 892y_{n+3}, \\ 4278y_{2n+4} - 205252y_{2n+3} + 2530 \end{array} \right)$

Some special cases among the solutions are given below:

$$P_y^{10}(t_{3,x+1})^2 = 207P_x^6(t_{3,y})^2 + 11(t_{3,y})^2(t_{3,x+1})^2$$

$$9P_y^6(t_{3,x})^2 = 23P_x^{10}(t_{3,y+1})^2 + 11(t_{3,x})^2(t_{3,y+1})^2$$

$$P_y^{10}(t_{3,2x-2})^2 = 23(6P_{x-1}^4)^2(t_{3,y})^2 + 11(t_{3,y})^2(t_{3,2x-2})^2$$

$$36P_{y-1}^8(t_{3,x})^2 = 23P_x^{10}(t_{3,2y-2})^2 + 11(t_{3,x})^2(t_{3,2y-2})^2$$

$$9P_y^6(t_{3,2x-2})^2 = 23(36P_{x-1}^8)(t_{3,y+1})^2 + 11(t_{3,2x-2})^2(t_{3,y+1})^2$$

$$(6P_{y-1}^4)^2(t_{3,x+1})^2 = 23(3P_x^3)^2(t_{3,2y-2})^2 + 11(t_{3,x+1})^2(t_{3,2y-2})^2$$

### III. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations  $y^2 = 23x^2 - 11$ . As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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