

Analytical Approaches to the Positive Pell Equation $x^2 = 6y^2 + 3^t$

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ABSTRACT

The binary quadratic Diophantine equation represented by the positive Pellian $x^2 = 6y^2 + 3^t$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-10]. In this communication, yet another interesting equation given by $x^2 = 6y^2 + 3^t$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$x^2 = 6y^2 + 3^t \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 3 \cdot 3^{t-1}, y_0 = 1 \cdot 3^{t-1}$$

The Pellian equation is

$$x^2 = 6y^2 + 1 \tag{2}$$

The initial solution of Pellian equation is

$$\tilde{x}_0 = 5, \tilde{y}_0 = 2$$

The general solution (x_n, y_n) of (2) is given by,

$$\tilde{x}_n = \frac{1}{2} f_n, \tilde{y}_n = \frac{1}{2\sqrt{6}} g_n$$

Where,

$$f_n = (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1}$$

$$g_n = (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{3 \cdot 3^{t-1}}{2} f_n + \frac{6 \cdot 3^{t-1}}{2\sqrt{6}} g_n$$

$$y_{n+1} = \frac{3^{t-1}}{2} f_n + \frac{3 \cdot 3^{t-1}}{2\sqrt{6}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0 \quad n=0,1,2,3,\dots$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

| n | x_n | y_n |
|-----|----------------------|----------------------|
| 0 | $3 \cdot 3^{t-1}$ | $1 \cdot 3^{t-1}$ |
| 1 | $27 \cdot 3^{t-1}$ | $11 \cdot 3^{t-1}$ |
| 2 | $267 \cdot 3^{t-1}$ | $109 \cdot 3^{t-1}$ |
| 3 | $2643 \cdot 3^{t-1}$ | $1079 \cdot 3^{t-1}$ |

From the above table, we observe some interesting relations among the solutions which are presented below.

Both x_n and y_n values are odd.

Each of the following expression is a nasty number

$$\frac{6}{3^t} [2 \cdot 3^t + 11x_{2n+2} - x_{2n+3}]$$

$$\frac{3}{10 \cdot 3^t} [40 \cdot 3^t + 218x_{2n+2} - 2x_{2n+4}]$$

$$\frac{6}{5 \cdot 3^t} [10 \cdot 3^t + 54x_{2n+2} - 12y_{2n+3}]$$

$$\frac{6}{49 \cdot 3^t} [98 \cdot 3^t + 534x_{2n+2} - 12y_{2n+4}]$$

$$\frac{6}{5 \cdot 3^t} [10 \cdot 3^t + 6x_{2n+3} - 132y_{2n+2}]$$

$$\frac{6}{49 \cdot 3^t} [98 \cdot 3^t + 6x_{2n+4} - 1308y_{2n+2}]$$

$$\frac{6}{3^t} [2 \cdot 3^t + 3y_{2n+3} - 27y_{2n+2}]$$

$$\frac{3}{10 \cdot 3^t} [40 \cdot 3^t + 6y_{2n+4} - 534y_{2n+4}]$$

$$\frac{1}{3^t} [12 \cdot 3^t + 654x_{2n+3} - 66x_{2n+4}]$$

$$\frac{6}{3^t} [2 \cdot 3^t + 54x_{2n+3} - 132y_{2n+3}]$$

$$\frac{6}{5 \cdot 3^t} [10 \cdot 3^t + 534x_{2n+3} - 132y_{2n+4}]$$

$$\frac{6}{5 \cdot 3^t} [10 \cdot 3^t + 54x_{2n+4} - 1308y_{2n+3}]$$

$$\frac{6}{3^t} [2 \cdot 3^t + 534x_{2n+4} - 1308y_{2n+4}]$$

$$\frac{6}{3^t} [2 \cdot 3^t + 27y_{2n+4} - 267y_{2n+3}]$$

Each of the following expressions is a cubical integer.

$$\frac{1}{3^t} [11x_{3n+3} - x_{3n+4} + 33x_{n+1} - 3x_{n+2}]$$

$$\frac{1}{10 \cdot 3^t} [109x_{3n+3} - x_{3n+5} + 327x_{n+1} - 3x_{n+2}]$$

$$\frac{1}{5 \cdot 3^t} [54x_{3n+3} - 12y_{3n+4} + 162x_{n+1} - 36y_{n+2}]$$

$$\frac{1}{49 \cdot 3^t} [534x_{3n+3} - 12y_{3n+5} + 1602x_{n+1} - 36y_{n+3}]$$

$$\frac{1}{5 \cdot 3^t} [6x_{3n+4} - 132y_{3n+3} + 18x_{n+2} - 396y_{n+1}]$$

$$\frac{1}{49 \cdot 3^t} [6x_{3n+5} - 1308y_{3n+3} + 18x_{n+2} - 3924y_{n+1}]$$

$$\frac{1}{3^t} [3y_{3n+4} - 27y_{3n+3} + 9y_{n+2} - 81y_{n+1}]$$

$$\frac{1}{10 \cdot 3^t} [3y_{3n+5} - 267y_{3n+3} + 9y_{n+2} - 801y_{n+1}]$$

$$\frac{1}{3^t} [109x_{3n+4} - 11x_{3n+5} + 327x_{n+2} - 33x_{n+2}]$$

$$\frac{1}{3^t} [54x_{3n+4} - 132y_{3n+4} + 162x_{n+2} - 396y_{n+2}]$$

$$\frac{1}{5 \cdot 3^t} [534x_{3n+4} - 132y_{3n+5} + 1602x_{n+2} - 396y_{n+3}]$$

$$\frac{1}{5 \cdot 3^t} [54x_{3n+5} - 1308y_{3n+4} + 162x_{n+3} - 3924y_{n+2}]$$

$$\frac{1}{3^t} [534x_{3n+5} - 1308y_{3n+5} + 1602x_{n+3} - 3924y_{n+3}]$$

$$\frac{1}{3^t} [27y_{3n+5} - 267y_{3n+4} + 81y_{n+3} - 801y_{n+2}]$$

Each of the following expressions is a biquadratic integer.

$$\begin{aligned} & \frac{1}{3^t} [11x_{4n+4} - x_{4n+5} + 44x_{2n+2} - 4x_{2n+3} + 6.3^t] \\ & \frac{1}{10.3^t} [109x_{4n+4} - x_{4n+4} + 436x_{2n+2} - 4x_{2n+4} + 60.3^t] \\ & \frac{1}{5.3^t} [54x_{4n+4} - 12y_{4n+5} + 270x_{2n+2} - 60y_{2n+3} + 30.3^t] \\ & \frac{1}{49.3^t} [534x_{4n+4} - 12y_{4n+6} + 2136x_{2n+2} - 48y_{2n+4} + 294.3^t] \\ & \frac{1}{5.3^t} [6x_{4n+5} - 132y_{4n+4} + 24x_{2n+3} - 528y_{2n+2} + 30.3^t] \\ & \frac{1}{49.3^t} [6x_{4n+6} - 1308y_{4n+4} + 24x_{2n+4} - 5232y_{2n+2} + 294.3^t] \\ & \frac{1}{3^t} [3y_{4n+5} - 27y_{4n+4} + 12y_{2n+3} - 108y_{2n+2} + 6.3^t] \\ & \frac{1}{10.3^t} [3y_{4n+6} - 267y_{4n+4} + 12y_{2n+4} - 1068y_{2n+1} + 60.3^t] \\ & \frac{1}{3^t} [109x_{4n+5} - 11x_{4n+6} + 436x_{2n+3} - 44x_{2n+4} + 6.3^t] \\ & \frac{1}{3^t} [54x_{4n+5} - 132y_{4n+5} + 216x_{2n+3} - 528y_{2n+3} + 6.3^t] \\ & \frac{1}{5.3^t} [534x_{4n+5} - 132y_{4n+6} + 2136x_{2n+3} - 528y_{2n+4} + 30.3^t] \\ & \frac{1}{5.3^t} [54x_{4n+6} - 1308y_{4n+5} + 216x_{2n+4} - 5232y_{2n+3} + 30.3^t] \\ & \frac{1}{3^t} [534x_{4n+6} - 1308y_{4n+6} + 2136x_{2n+4} - 5232y_{2n+4} + 6.3^t] \\ & \frac{1}{3^t} [27y_{4n+6} - 267y_{4n+5} + 108y_{2n+4} + 106y_{2n+3} + 6.3^t] \end{aligned}$$

Each of the following expression is a quintic integer:

$$\begin{aligned} & \frac{1}{3^t} [6x_{5n+5} - 12y_{5n+5} + 30x_{3n+3} - 60y_{3n+3} + 60x_{n+1} - 120y_{n+1}] \\ & \frac{1}{3^t} [11x_{5n+5} - x_{5n+6} + 55x_{3n+3} - 5x_{3n+4} + 110x_{n+1} - 10x_{n+2}] \\ & \frac{1}{5.3^t} [109x_{5n+5} - x_{5n+7} + 545x_{3n+3} - 5x_{3n+5} + 1090x_{n+1} - 10x_{n+2}] \\ & \frac{1}{5.3^t} [54x_{5n+5} - 12y_{5n+6} + 270x_{3n+3} - 60x_{3n+4} - 540x_{n+1} - 120y_{n+2}] \end{aligned}$$

$$\begin{aligned} & \frac{1}{49.3^t} [534x_{5n+5} - 12y_{5n+7} + 2670x_{3n+3} - 60y_{3n+5} + 5340x_{n+1} - 120y_{n+3}] \\ & \frac{1}{5.3^t} [6x_{5n+6} - 132y_{5n+5} + 30x_{3n+4} - 660y_{3n+3} + 60x_{n+2} - 1320y_{n+1}] \\ & \frac{1}{49.3^t} [6x_{5n+7} - 1308y_{5n+5} + 30x_{3n+5} - 6540y_{3n+3} + 60x_{n+2} - 13080y_{n+1}] \\ & \frac{1}{3^t} [3y_{5n+6} - 27y_{5n+5} + 15y_{3n+4} - 135y_{3n+3} + 30y_{n+2} - 270y_{n+1}] \\ & \frac{1}{10.3^t} [3y_{5n+7} - 267y_{5n+5} + 15y_{3n+5} - 1335y_{3n+3} + 30y_{n+2} - 2670y_{n+1}] \\ & \frac{1}{3^t} [109x_{5n+6} - 11x_{5n+7} + 545x_{3n+4} - 55x_{3n+5} + 1090x_{n+2} - 110x_{n+3}] \\ & \frac{1}{3^t} [54x_{5n+6} - 132y_{5n+6} + 270x_{3n+4} - 660y_{3n+4} + 540y_{n+2} - 1320x_{n+2}] \\ & \frac{1}{5.3^t} [534x_{5n+6} - 132y_{5n+7} + 2670x_{3n+4} - 660y_{3n+5} + 5340x_{n+2} - 1320y_{n+3}] \\ & \frac{1}{5.3^t} [54x_{5n+7} - 1308y_{5n+6} + 270x_{3n+5} - 6540y_{3n+4} + 540x_{n+3} - 13080y_{n+2}] \\ & \frac{1}{3^t} [27y_{5n+7} - 267y_{5n+6} + 135y_{3n+5} - 1335y_{3n+4} + 270y_{n+3} - 2670y_{n+2}] \\ & \frac{1}{3^t} [534x_{5n+7} - 1308y_{5n+7} + 2670x_{3n+5} - 6540y_{3n+5} + 5340x_{n+3} - 13080y_{n+3}] \end{aligned}$$

Relations among the solutions are given below:

$$\begin{aligned} x_{n+3} &= 49x_{n+1} + 120y_{n+1} \\ y_{n+2} &= 2x_{n+1} + 5y_{n+1} \\ y_{n+3} &= 20x_{n+1} + 49y_{n+1} \\ x_{n+3} &= 10x_{n+2} - x_{n+1} \\ 12y_{n+2} &= 5x_{n+2} - x_{n+1} \\ 12y_{n+3} &= 49x_{n+2} - 5x_{n+1} \\ 24y_{n+2} &= x_{n+3} - x_{n+1} \\ 120y_{n+3} &= 49x_{n+3} - x_{n+1} \\ 5y_{n+3} &= 49y_{n+2} + 2x_{n+1} \\ 5x_{n+3} &= 12y_{n+1} + 49x_{n+2} \\ 5y_{n+2} &= y_{n+1} + 2x_{n+2} \\ y_{n+3} &= y_{n+1} + 4x_{n+2} \\ 49y_{n+2} &= 5y_{n+1} + 2x_{n+3} \\ 49y_{n+3} &= y_{n+1} + 20x_{n+3} \end{aligned}$$

$$y_{n+3} = 10y_{n+2} - y_{n+1}$$

$$12y_{n+2} = x_{n+3} - 5x_{n+2}$$

$$12y_{n+3} = 5x_{n+3} - x_{n+2}$$

$$y_{n+3} = 5y_{n+2} + 2x_{n+2}$$

$$5y_{n+3} = y_{n+2} + 2x_{n+3}$$

Remarkable Observation

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

| S.NO | Hyperbola | (X,Y) |
|------|-------------------------|-------------------------------------------------------|
| 1 | $X^2 - 6Y^2 = 4.9^t$ | $(6x_{n+1} - 12y_{n+1}, 6y_{n+1} - 2x_{n+1})$ |
| 2 | $X^2 - 6Y^2 = 4.9^t$ | $(11x_{n+1} - x_{n+2}, 2x_{n+2} - 18x_{n+1})$ |
| 3 | $X^2 - 6Y^2 = 1600.9^t$ | $(218x_{n+1} - 2x_{n+3}, x_{n+3} - 89x_{n+1})$ |
| 4 | $X^2 - 6Y^2 = 100.9^t$ | $(54x_{n+1} - 12y_{n+2}, 6y_{n+2} - 22x_{n+1})$ |
| 5 | $X^2 - 6Y^2 = 9604.9^t$ | $(534x_{n+1} - 12y_{n+3}, 6y_{n+3} - 218x_{n+1})$ |
| 6 | $X^2 - 6Y^2 = 100.9^t$ | $(6x_{n+2} - 132y_{n+1}, 54y_{n+1} - 2x_{n+2})$ |
| 7 | $X^2 - 6Y^2 = 9604.9^t$ | $(6x_{n+3} - 1308y_{n+1}, 534y_{n+1} - 2x_{n+3})$ |
| 8 | $X^2 - 6Y^2 = 4.9^t$ | $(3y_{n+2} - 27y_{n+1}, 11y_{n+1} - y_{n+2})$ |
| 9 | $X^2 - 6Y^2 = 1600.9^t$ | $(6y_{n+3} - 534y_{n+1}, 218y_{n+1} - 2y_{n+3})$ |
| 10 | $X^2 - 6Y^2 = 144.9^t$ | $(654x_{n+2} - 66x_{n+3}, 27x_{n+3} - 267x_{n+2})$ |
| 11 | $X^2 - 6Y^2 = 4.9^t$ | $(54x_{n+2} - 132y_{n+2}, 54y_{n+2} - 22x_{n+2})$ |
| 12 | $X^2 - 6Y^2 = 100.9^t$ | $(534x_{n+2} - 132y_{n+3}, 54y_{n+3} - 218x_{n+2})$ |
| 13 | $X^2 - 6Y^2 = 100.9^t$ | $(54x_{n+3} - 1308y_{n+2}, 534y_{n+2} - 22x_{n+3})$ |
| 14 | $X^2 - 6Y^2 = 4.9^t$ | $(534x_{n+3} - 1308y_{n+3}, 534y_{n+3} - 218x_{n+3})$ |
| 15 | $X^2 - 6Y^2 = 4.9^t$ | $(27y_{n+3} - 267y_{n+2}, 109y_{n+2} - 11y_{n+3})$ |

Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

| S.NO | Parabola | (X,Y) |
|------|--------------------------|-----------------------------------------------------------------------------------------------------|
| 1 | $3^t X^2 - 6Y^2 = 4.9^t$ | $\left(\begin{matrix} 6x_{2n+2} - 12y_{2n+2} + 2.3^t, \\ 6y_{n+1} - 2x_{n+1} \end{matrix} \right)$ |

| | | |
|----|--------------------------------|-------------------------------------------------------------------------------------------------------------|
| 2 | $3^t X^2 - 6Y^2 = 4.9^t$ | $\left(\begin{matrix} 11x_{2n+2} - x_{2n+3} + 2.3^t, \\ 2x_{n+2} - 18x_{n+1} \end{matrix} \right)$ |
| 3 | $20.3^t X^2 - 6Y^2 = 1600.9^t$ | $\left(\begin{matrix} 218x_{2n+2} - 2x_{2n+4}, \\ x_{n+3} - 89x_{n+1} \end{matrix} \right)$ |
| 4 | $5.3^t X^2 - 6Y^2 = 100.9^t$ | $\left(\begin{matrix} 54x_{2n+2} - 12y_{2n+3} + 10.3^t, \\ 6y_{n+2} - 22x_{n+1} \end{matrix} \right)$ |
| 5 | $49.3^t X^2 - 6Y^2 = 9604.9^t$ | $\left(\begin{matrix} 534x_{2n+2} - 12y_{2n+4} + 98.3^t, \\ 6y_{n+3} - 218x_{n+1} \end{matrix} \right)$ |
| 6 | $5.3^t X^2 - 6Y^2 = 100.9^t$ | $\left(\begin{matrix} 6x_{2n+3} - 132y_{2n+2} + 10.3^t, \\ 54y_{n+1} - 2x_{n+2} \end{matrix} \right)$ |
| 7 | $49.3^t X^2 - 6Y^2 = 9604.9^t$ | $\left(\begin{matrix} 6x_{2n+4} - 1308y_{2n+2} + 98.3^t, \\ 534y_{n+1} - 2x_{n+3} \end{matrix} \right)$ |
| 8 | $4.9^t X^2 - 6Y^2 = 4.9^t$ | $\left(\begin{matrix} 3y_{2n+3} - 27y_{2n+2} + 2.3^t, \\ 11y_{n+1} - y_{n+2} \end{matrix} \right)$ |
| 9 | $20.3^t X^2 - 6Y^2 = 1600.9^t$ | $\left(\begin{matrix} 6y_{2n+4} - 534y_{2n+3} + 40.3^t, \\ 218y_{n+1} - 2y_{n+3} \end{matrix} \right)$ |
| 10 | $6.3^t X^2 - 6Y^2 = 144.9^t$ | $\left(\begin{matrix} 654x_{2n+3} - 66x_{2n+4} + 12.3^t, \\ 27x_{n+3} - 267x_{n+2} \end{matrix} \right)$ |
| 11 | $3^t X^2 - 6Y^2 = 4.9^t$ | $\left(\begin{matrix} 54x_{2n+3} - 132y_{2n+3} + 2.3^t, \\ 54y_{n+2} - 22x_{n+2} \end{matrix} \right)$ |
| 12 | $5.3^t X^2 - 6Y^2 = 100.9^t$ | $\left(\begin{matrix} 534x_{2n+3} - 132y_{2n+4} + 10.3^t, \\ 54y_{n+3} - 218x_{n+2} \end{matrix} \right)$ |
| 13 | $5.3^t X^2 - 6Y^2 = 100.9^t$ | $\left(\begin{matrix} 54x_{2n+4} - 1308y_{2n+3} + 10.3^t, \\ 534y_{n+2} - 22x_{n+3} \end{matrix} \right)$ |
| 14 | $3^t X^2 - 6Y^2 = 4.9^t$ | $\left(\begin{matrix} 534x_{2n+4} - 1308y_{2n+4} + 2.3^t, \\ 534y_{n+3} - 218x_{n+3} \end{matrix} \right)$ |
| 15 | $3^t X^2 - 6Y^2 = 4.9^t$ | $\left(\begin{matrix} 27y_{2n+4} - 267y_{2n+3} + 2.3^t, \\ 109y_{n+2} - 11y_{n+3} \end{matrix} \right)$ |

III. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $x^2 = 6y^2 + 3^t$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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