Black-Scholes Option Pricing Model: A Stochastic Differential Equation Approach

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ABSTRACT

The aim of this paper is to study Black-Scholes option pricing model using stochastic differential equation. This option pricing model is one of the biggest success in financial engineering both in terms of approach and applicability. We discuss some useful definitions and terminologies which are useful in the development of this model. We also discuss Ito's process which plays important role in this option pricing model.We study the derivation of this option pricing model using stochastic differential equation. As an application we study the model with its parameter using Maple software.

Keywords: *Black-Scholes Option pricing model, Call Option, Maple, Option price, Volatility*.

I. INTRODUCTION

Financial Engineering is one of the new academic discipline that has variety of different knowledge areas such as physics, mathematics, computer science and finance. It uses concepts from knowledge areas to achieve a deeper understanding of the price dynamics of the individual securities, portfolios, and the financial markets (Hull et al. ,2005).Study of option pricing model involves financial processes such as stock prices, interest rates, exchange rates and pricing derivatives on basis underlying asset (Ross Sheldon, 1999). In the year 1973, Fischer Black and Myron Scholes develop the original option pricing formula and it is published in the paper entitled "The Pricing of Options and Corporate Liabilities" in the journal of political economy (F. Black et al. , 1973). The black-Scholes models governs the price of the option over time. Due to its simplicity and clarity to obtain the price option calls, the Black-Scholes option pricing model is used in financial engineering very often. In the year 1973, Fisher Black and Myron Scholes won the Nobel Prize of Economics for this model. The model assumes the option price follows a geometric Brownian motion with constant drift and volatility (Ingmar Glauche et al. ,2001).

In our study, we discuss some basic definitions, derivations and lemma's which are useful in the development of Black-Scholes option pricing model. Also, we use the maple software to obtain the solution of Black-Scholes equation by varying it parameters and these are represented graphically.

We organize the paper as follows: The section 2, is devoted for some definitions and terminologies. In section 3, we study Ito process. We study the derivation of the Black-Scholes model using stochastic differential equation, in section 3. In the last section, we derive the solution of Black-Scholes equation with valuing an option using Maple software.

II. DEFINITIONS AND TERMINOLOGIES

2.1: Option

A security giving the right to buy or sell an asset, subject to certain conditions, within a specified period of time is called as an option (Wilmott et al. , 1997).

There are two types of options which are defined as follows:

(i) **Call Option**: An option which grants its holder the right to buy the underlying asset at a strike price at some moment in the future is called as call option.

(ii) **Put Option**: An option which grants its holder the right to sell the underlying asset at a strike price at some moment in the future is called as put option.

2.2: Expiration Date

The date on which an option right or warrant expires, and becomes worthless if not exercised is called an expiration date. There are two different types of options with respect to expiration.

(i) **European Option**: An option which cannot be exercised until the expiration date is called an European Option.

(ii) **American Option**: An option which can be exercised at any time up to and including the expiration date is called as an American Option.

2.3: Risk-Less Interest Rate
The annual interest rate of bonds or other “risk-free” investments, is called as the risk-less interest rate. It is denoted by r.

2.4: Volatility
A measure for variation of price of a financial instrument over time is called volatility (Baxter, 1996). There are two important types of volatility as follows:
(i) **Implied Volatility**: Volatility derived from the market price of a market traded derivative is called implied volatility.
(ii) **Historic Volatility**: Volatility derived from time series of past market prices is called historic volatility.

2.5: Strike Price
The predetermined price of an underlying asset is called as strike price.

2.6: Portfolio
Any collection of financial assets such as stocks, bonds and cash equivalents held by an investment institution or company is called portfolio.

2.7: Geometric Brownian Motion
A continuous time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion is called geometric Brownian Motion (Klebaner et al. ,1999).

2.9: Stochastic Differential Equation
Let (Ω, F, P) be a probability space and let X_t, t, ε R_+ be a stochastic process X:Ω x R_t→ R. Moreover, assume that a (X_t, t):Ω x R x R_+→ R and b(X_t, t):Ω x R x R_+→ R are stochastically integrable functions of t ε R_+. Then the equation

$$dX_t = a\,(X_t, t)dt + b(X_t, t)dW_t \qquad (1)$$

is called stochastic differential equation.
Note that (1) has been understood as a symbolic notation of the stochastic integral equation.

$$X_t = X_0 + \int_0^t a\,(X_s\,, S)ds + \int_0^t b\,(X_s\,, S)dWs \qquad (2)$$

The function $a\,(X_t\,, t)$ and $b\,(X_t\,, t)$ are referredto as the drift term and the diffusion term respectively.

III. ITÔ PROCESS

A stochastic process $X_t$satisfying equation

$$dX_t = a\,(X_t\,, t)dt + b(X_t\,, t)dW_t$$

is said to be an Itô process (Baxer et al. , 1996).

3.1**: Itô Integral**
Assume that b = $b(t)$ is a stochastically integrable function in the sense that there exists a sequence b_n, n ε N of simple processes such that

$$\lim_{n\to\infty} E(\int_0^T \left(b(t) - b_n(t)\right)^2 dt\,) = 0$$

Then, the Itô integral of b is defined as

$$\int_0^T b\,(t)dW_t = \lim_{n\to\infty} \int_0^T b_n(t)dW_t$$

3.2: Itô Lemma

Let X_t, t ε R₊, be an Itô process X : Ω x R₊→ R and $f := C^2\ (R\ x\ R_+\ x\ R_+)$. Then, the stochastic process $f_t := f(X_t\ , t)$is also an Itô process which satisfies

$$df_t = \left(\frac{\partial f}{\partial t} + a\ \frac{\partial f}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial t^2}\right) dt + \frac{\partial f}{\partial x} dW_t \quad (3)$$

Proof: By Taylor's series the expansion of $f(X_{t+\Delta t}, t + \Delta t)$about $(X_t\ , t)$is given as follows

$$f(X_{t+\Delta t}, t + \Delta t) = f(X_t, t) + \frac{\partial f}{\partial t}(X_t, t)(\Delta t) + \frac{\partial f}{\partial x}(X_t, t)(X_{t+\Delta t} - X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(X_t, t)(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(X_t, t)(X_{t+\Delta t}, t - X_t)^2 + \frac{\partial^2 f}{\partial x \partial t}(X_t, t)(\Delta t)(X_{t+\Delta t} - X_t) + O(\Delta t)^2 + O(\Delta t)(X_{t+\Delta t} - X_t)^2 + O((X_{t+\Delta t} - X_t)^3)$$

Taking, limit as Δt→0 gives

$$df_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} dX_t{}^2 + O((\mathrm{d}t)^2) + O(dt(\mathrm{d}X_t)^2 + O\ ((dX_t)^3) \quad (4)$$

Consider $X_t$is anItô Process and $dW_t{}^2 = dt$, then from equation (4), we get

$$dX_t{}^2 = (adt + bdW_t)^2 = a^2(dt)^2 + 2abdtdW_t + b^2 dW_t{}^2$$
$$= b^2 + O((dt)^{3/2}) \quad (5)$$

From equation (4) and (5), we get

$$df_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x}(adt + bdW_t) + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial t^2} dt$$
$$df_t = (\frac{\partial f}{\partial t} + a\ \frac{\partial f}{\partial x} + \frac{1}{2} b^2\ \frac{\partial^2 f}{\partial x^2})dt + b\frac{\partial f}{\partial x} dW_t (6)$$

where W_t is a Wiener process.

IV. BLACK-SCHOLES OPTION PRICING MODEL USING STOCHASTIC DIFFERENTIAL EQUATION.

This option price model is designed to price an option as a function of certain variables generally stock price, striking price, volatility, time to expiration, dividends to be paid and the current risk-free interest rate for pricing European Options on stocks, developed by Fisher Black, Myron Scholes and Robert Merton.

In this section, first we discuss some basic assumptions underlying the Black-Scholes model of calculating options pricing. The most significant is that volatility, a measure of how much a stock can be expected to move in the near-term, is a constant over time. The Black-Scholes model also assumes stocks move in a manner referred to as an random walk; at any given moment, they are as likely to move up as they are to move down. These assumptions are combined with the principle that options pricing should provide no immediate gain to either seller or buyer. The exact six assumptions of Black-Scholes models are as follows:

(i) stock pays no dividend,
(ii) option can only be exercised upon expiration,
(iii) market direction cannot be predicted, hence "Random walk",
(iv) no commissions are charged in the transaction,
(v) interest rated remain constant
(vi) stock returns are normally distributed, thus volatility is constant over time

We obtain the derivation of the Black-Scholes option pricing model using the stochastic differential equation and Itôs Lemma. We first consider the price process for assets. It is obtain by a stochastic differential equation based on a geometric Brownian motion (Wilmott et al. 1997). Then change or price difference in the asset prices is assumed to be a Markow process. The return is, the change in the price divided by its original value. If v is the average rate of growth of the asset price, then the deterministic contribution in time dt is vdt. If σ is the volatility related to the standard deviation of the returns and dX a sample from a normal distribution, then the contribution is assumed to be σdX. Therefore, the resulting equation can be written in the following form:

Consider a general option values V(S, t). Therefore, from Taylor's theorem we have the following series expansion for V(S, t):

$$\frac{dS}{S} = vdt + \sigma dX \tag{7}$$

The equation (7) is called the stochastic differential equation (Brzezniak et al.,1998). The normal distribution used in (7) is a Weiner process with the following properties:

$$(i) E(dX) = 0,$$
$$(ii) E(dX)^2 = dt$$

Now, σ is proportional to $Var(dS)$, the expectation and variance are calculated as follows.

$$E(dS) = E(vS\,dt + \sigma S\,dX) = E(vS\,dt) + E(\sigma S\,dX)$$
$$= vS\,dt\; Var(dS)$$
$$= E(dS^2) - [E(dS)]^2$$
$$= E[(vS\,dt + \sigma S\,dX)^2] - (vS\,dt)^2$$
$$= \sigma^2 S^2 dX^2$$

Since E $(S^2 dX dt) = 0$, the standard deviation is the square-root of the variance. Therefore, σ is proportional to

$$\frac{\sqrt{Var(ds)}}{S}$$

This is the Stochastic model.

V. APPLICATIONS

As an application of Black-Scholes model, we obtain the solution of Black-Scholes equation using the Maple software. In our test problem, we chose the parameters as follows:
E = 49.9, T = 0.5, r = 0.10, σ = 0.40

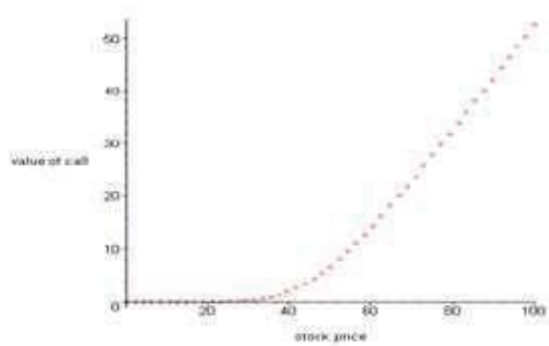

Figure 4.1: The solution for a European call with parameters: E = 50

VI. CONCLUSIONS

1) We study Ito's process and stochastic differential equation.
2) We study the Black-Scholes option pricing model and it limitations.
3) The Black Scholes partial differential equation is very much useful in the financial engineering.
4) We obtain the call option values of an underlying asset by Black-Scholes equation and these are simulated by Maple software.

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REFERENCES

1. *Baxter, Martin and Andrew Rennie (1996), "Financial Calculus: An introduction to derivative Pricing", Cambridge, England,Cambridge University Press.*
2. *Brzezniak, Z, Zastawniak, T (1998), "Basic Stochastic Processes",Springer-Verlag, Heidelberg.*
3. *F. Black and M. Scholes (1973), "The pricing of Options and Corporate Liabilities", Journal of Political Economy.*
4. *Hull, John C (2005),"Options, Futures and Other Derivatives", Prentice Hall India.*
5. *Ingmar Glauche (2001), "A classical Approach to the Black and Scholes Formula and its Critiques, Discretization of the model".*
6. *Klebaner, Fima C (1999), "Introduction to stochastic Calculus with Applications", London, Imperial College Press.*
7. *Ross Sheldon (1999), "An introduction to Mathematical Finance", Cambridge, England, Cambridge University Press.*
8. *Wilmott, P. Howison, S, Dewynne, J. (1997), "The Mathematics of Financial Derivatives", Cambridge University Press, Cambridge.*