EXPLORING SOLUTIONS OF THE BINARY QUADRATIC EQUATION $y^2 = 35x^2 + 29$

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ABSTRACT

The binary quadratic equation represented by the positive pellian $y^2 = 35x^2 + 29$ is analyzed for its distinct integer solutions. A few interesting relation among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas

KEYWORDS: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

1. INTRODUCTION

The binary quadratic Diophantine equations are rich in variety. The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non square positive integer has been satisfied by various mathematician for its non-trivial integral solution. When D takes different integral values [1-4]. In [5-15] the binary quadratic non-homogeneous equation representing hyperbolas respectively are studied for their non-zero integral solutions. This communication concerns with yet another binary quadratic equation given by $y^2 = 35x^2 + 29$.The recurrence relation satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is

$$y^2 = 35x^2 + 29 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1,\ y_0 = 8$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 35x^2 + 1$$

whose smallest positive integer solution is

$$\tilde{x}_0 = 1,\ \tilde{y}_0 = 6$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{35}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(6+\sqrt{35}\right)^{n+1} + \left(6-\sqrt{35}\right)^{n+1}$$

$$g_n = \left(6+\sqrt{35}\right)^{n+1} - \left(6-\sqrt{35}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{4}{\sqrt{35}} g_n$$

$$y_{n+1} = 4 f_n + \frac{\sqrt{35}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 12x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 12y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	8
0	14	83
1	167	988
2	1990	11773
3	23713	140288
4	282566	1671683

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} & y_{n+1} values are alternatively odd and even .

2. Relations among the solutions

- $x_{n+3} - 12x_{n+2} + x_{n+1} = 0$
- $y_{n+1} - x_{n+2} + 6x_{n+1} = 0$
- $y_{n+2} - 6x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 71x_{n+2} + 6x_{n+1} = 0$
- $12y_{n+1} + 71x_{n+1} - x_{n+3} = 0$
- $2y_{n+2} + x_{n+1} - x_{n+3} = 0$
- $12y_{n+3} + x_{n+1} - 71x_{n+3} = 0$
- $y_{n+2} - 6y_{n+1} - 35x_{n+1} = 0$
- $y_{n+3} - 71y_{n+1} - 420x_{n+1} = 0$
- $6y_{n+1} - y_{n+2} + 35x_{n+1} = 0$
- $6y_{n+3} - 71y_{n+2} - 35x_{n+1} = 0$
- $y_{n+1} + 71x_{n+2} - 6x_{n+3} = 0$
- $y_{n+2} + 6x_{n+1} - x_{n+3} = 0$
- $y_{n+3} + x_{n+2} - 6x_{n+3} = 0$
- $6y_{n+2} - y_{n+1} - 35x_{n+2} = 0$
- $6x_{n+2} - x_{n+3} + y_{n+2} = 0$
- $12y_{n+2} - y_{n+1} - y_{n+3} = 0$
- $35x_{n+1} + 71y_{n+2} - 6y_{n+3} = 0$
- $35x_{n+3} + y_{n+2} - 6y_{n+3} = 0$
- $y_{n+1} + 420x_{n+3} - 71y_{n+3} = 0$

3. Each of the following expressions represents a nasty number

- $\frac{1}{29}\left(96x_{2n+3} - 996x_{2n+2} + 348\right)$
- $\frac{1}{87}\left(24x_{2n+4} - 2964x_{2n+2} + 1044\right)$
- $\frac{1}{29}\left(96y_{2n+2} - 420x_{2n+2} + 348\right)$

- $\frac{1}{87}\left(48y_{2n+3}-2940x_{2n+2}+1044\right)$
- $\frac{1}{2059}\left(96y_{2n+4}-70140x_{2n+2}+24708\right)$
- $\frac{1}{29}\left(996x_{2n+4}-11856x_{2n+3}+348\right)$
- $\frac{1}{87}\left(498y_{2n+2}-210x_{2n+3}+1044\right)$
- $\frac{1}{29}\left(996y_{2n+3}-5880x_{2n+3}+348\right)$
- $\frac{1}{87}\left(498y_{2n+4}-35070x_{2n+3}+1044\right)$
- $\frac{1}{2059}\left(11856y_{2n+2}-420x_{2n+4}+24708\right)$
- $\frac{1}{87}\left(5928y_{2n+3}-2940x_{2n+4}+1044\right)$
- $\frac{1}{29}\left(11856y_{2n+4}-70140x_{2n+4}+348\right)$
- $\frac{1}{29}\left(168y_{2n+2}-12y_{2n+3}+348\right)$
- $\frac{1}{87}\left(501y_{2n+2}-3y_{2n+4}+1044\right)$
- $\frac{1}{29}\left(2004y_{2n+3}-168y_{2n+4}+348\right)$

4. Each of the following expressions represents a cubical integer

- $\frac{1}{29}\left[16x_{3n+4}-166x_{3n+3}+48x_{n+2}-498x_{n+1}\right]$
- $\frac{1}{87}\left[4x_{3n+5}-494x_{3n+3}+12x_{n+3}-4482x_{n+1}\right]$
- $\frac{1}{29}\left[16y_{3n+3}-70x_{3n+3}+48y_{n+1}-210x_{n+1}\right]$
- $\frac{1}{87}\left[8y_{3n+4}-490x_{3n+3}+24y_{n+2}-1470x_{n+1}\right]$
- $\frac{1}{2059}\left[16y_{3n+5}-11690x_{3n+3}+48y_{n+3}-35070x_{n+1}\right]$
- $\frac{1}{29}\left[166x_{3n+5}-1976x_{3n+4}+498x_{n+3}-5928x_{n+2}\right]$
- $\frac{1}{87}\left[83y_{3n+3}-35x_{3n+4}+249y_{n+1}-105x_{n+2}\right]$
- $\frac{1}{29}\left[166y_{3n+4}-980x_{3n+4}+498y_{n+2}-2940x_{n+2}\right]$
- $\frac{1}{87}\left[83y_{3n+5}-5845x_{3n+4}+249y_{n+3}-17535x_{n+2}\right]$

- $\frac{1}{2059}\left[1976y_{3n+3}-70x_{3n+5}+5928y_{n+1}-210x_{n+3}\right]$
- $\frac{1}{87}\left[988y_{3n+4}-490x_{3n+5}+2964y_{n+2}-1470x_{n+3}\right]$
- $\frac{1}{29}\left[1976y_{3n+5}-11690x_{3n+5}+5928y_{n+3}-35070x_{n+3}\right]$
- $\frac{1}{29}\left[28y_{3n+3}-2y_{3n+4}+84y_{n+1}-6y_{n+2}\right]$
- $\frac{1}{174}\left[167y_{3n+3}-y_{3n+5}+501y_{n+1}-3y_{n+3}\right]$
- $\frac{1}{29}\left[334y_{3n+4}-28y_{3n+5}+1002y_{n+2}-84y_{n+3}\right]$

5. Each of the following expressions represents a bi-quadratic integer

- $\frac{1}{29}\left[16x_{4n+5}-166x_{4n+4}+64x_{2n+3}-664x_{2n+2}+174\right]$
- $\frac{1}{87}\left[4x_{4n+6}-494x_{4n+4}+16x_{2n+4}-1976x_{2n+2}+522\right]$
- $\frac{1}{29}\left[16y_{4n+4}-70x_{4n+4}+64y_{2n+2}-280x_{2n+2}+174\right]$
- $\frac{1}{87}\left[8y_{4n+5}-490x_{4n+4}+32y_{2n+3}-1960x_{2n+2}+522\right]$
- $\frac{1}{2059}\left[16y_{4n+6}-11690x_{4n+4}+64y_{2n+4}-46760x_{2n+2}+12354\right]$
- $\frac{1}{29}\left[166x_{4n+6}-1976x_{4n+5}+664x_{2n+4}-7904x_{2n+3}+174\right]$
- $\frac{1}{87}\left[83y_{4n+4}-35x_{4n+5}+140x_{2n+3}-332y_{2n+2}+522\right]$
- $\frac{1}{29}\left[166y_{4n+5}-980x_{4n+5}+664y_{2n+3}-3920x_{2n+3}+174\right]$
- $\frac{1}{87}\left[83y_{4n+6}-5845x_{4n+5}+332y_{2n+4}-23380x_{2n+3}+522\right]$
- $\frac{1}{2059}\left[1976y_{4n+4}-70x_{4n+6}+7904y_{2n+2}-280x_{2n+4}+12354\right]$
- $\frac{1}{87}\left[988y_{4n+5}-490x_{4n+6}+3952y_{2n+3}-1960x_{2n+4}+522\right]$
- $\frac{1}{29}\left[1976y_{4n+6}-11690x_{4n+6}+7904y_{2n+4}-46760x_{2n+4}+174\right]$
- $\frac{1}{29}\left[28y_{4n+4}-2y_{4n+5}+112y_{2n+2}-8y_{2n+3}+174\right]$
- $\frac{1}{174}\left[167y_{4n+4}-y_{4n+6}+668y_{2n+2}-4y_{2n+4}+1044\right]$
- $\frac{1}{29}\left[334y_{4n+5}-28y_{4n+6}+1336y_{2n+3}-112y_{2n+4}+174\right]$

6. Each of the following expressions represents a quintic integer

- $\frac{1}{29}\left[16x_{5n+6}-166x_{5n+5}+80x_{3n+4}-830x_{3n+3}+160x_{n+2}-1660x_{n+1}\right]$
- $\frac{1}{87}\left[4x_{5n+7}-494x_{5n+5}+20x_{3n+5}-2470x_{3n+3}+40x_{n+3}-4940x_{n+1}\right]$
- $\frac{1}{29}\left[16y_{5n+5}-70x_{5n+5}+80y_{3n+3}-350x_{3n+3}+160y_{n+1}-700x_{n+1}\right]$
- $\frac{1}{87}\left[8y_{5n+6}-490x_{5n+5}+40y_{3n+4}-2450x_{3n+3}+80y_{n+2}-4900x_{n+1}\right]$
- $\frac{1}{2059}\left[16y_{5n+7}-11690x_{5n+5}+80y_{3n+5}-58450x_{3n+3}+160y_{n+3}-116900x_{n+1}\right]$
- $\frac{1}{29}\left[166x_{5n+7}-1976x_{5n+6}+830x_{3n+5}-9880x_{3n+4}+1660x_{n+3}-19760x_{n+2}\right]$
- $\frac{1}{87}\left[83y_{5n+5}-35x_{5n+6}+415y_{3n+3}-175x_{3n+4}+830y_{n+1}-350x_{n+2}\right]$
- $\frac{1}{29}\left[166y_{5n+6}-980x_{5n+6}+830y_{3n+4}-4900x_{3n+4}+1660y_{n+2}-9800x_{n+2}\right]$
- $\frac{1}{87}\left[83y_{5n+7}-5845x_{5n+6}+415y_{3n+5}-29225x_{3n+4}+830y_{n+3}-58450x_{n+2}\right]$
- $\frac{1}{2059}\left[1976y_{5n+5}-70x_{5n+7}+9880y_{3n+3}-350x_{3n+5}+19760y_{n+1}-700x_{n+3}\right]$
- $\frac{1}{87}\left[988y_{5n+6}-490x_{5n+7}+4940y_{3n+4}-2450x_{3n+5}+9880y_{n+2}-4900x_{n+3}\right]$
- $\frac{1}{29}\left[1976y_{5n+7}-11690x_{5n+7}+9880y_{3n+5}-58450x_{3n+5}+19760y_{n+3}-116900x_{n+3}\right]$
- $\frac{1}{29}\left[28y_{5n+5}-2y_{5n+6}-10y_{3n+4}+140y_{3n+3}+280y_{n+1}-20y_{n+2}\right]$
- $\frac{1}{174}\left[167y_{5n+5}-y_{5n+7}-5y_{3n+5}+835y_{3n+3}-10y_{n+3}+1670y_{n+1}\right]$
- $\frac{1}{29}\left[334y_{5n+6}-28y_{5n+7}+1670y_{3n+5}-140y_{3n+4}+3340y_{n+2}-28y_{n+3}\right]$

Remarkable observations:

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbola	(X,Y)
1	$X^2-35Y^2=841$	$(8x_{n+2}-83x_{n+1},14x_{n+1}-x_{n+2})$
2	$X^2-35Y^2=121104$	$(8x_{n+3}-988x_{n+1},167x_{n+1}-x_{n+3})$
3	$X^2-35Y^2=841$	$(8y_{n+1}-35x_{n+1},8x_{n+1}-y_{n+1})$
4	$X^2-35Y^2=30276$	$(8y_{n+2}-490x_{n+1},83x_{n+1}-y_{n+2})$
5	$X^2-35Y^2=4239481$	$(8y_{n+3}-5845x_{n+1},988x_{n+1}-y_{n+3})$

6	$X^2 - 35Y^2 = 841$	$(83x_{n+3} - 988x_{n+2}, 167x_{n+2} - 14x_{n+3})$
7	$X^2 - 35Y^2 = 30276$	$(83y_{n+1} - 35x_{n+2}, 8x_{n+2} - 14y_{n+1})$
8	$X^2 - 35Y^2 = 841$	$(83y_{n+2} - 490x_{n+2}, 83x_{n+2} - 14y_{n+2})$
9	$X^2 - 35Y^2 = 30276$	$(83y_{n+3} - 5845x_{n+2}, 988x_{n+2} - 14y_{n+3})$
10	$X^2 - 35Y^2 = 4239481$	$(988y_{n+1} - 35x_{n+3}, 8x_{n+3} - 167y_{n+1})$
11	$X^2 - 35Y^2 = 30276$	$(988y_{n+2} - 490x_{n+3}, 83x_{n+3} - 167y_{n+3})$
12	$X^2 - 35Y^2 = 841$	$(988y_{n+3} - 5845x_{n+3}, 988x_{n+3} - 167y_{n+3})$
13	$35X^2 - Y^2 = 29435$	$(14y_{n+1} - y_{n+2}, 8y_{n+2} - 83y_{n+1})$
14	$35X^2 - Y^2 = 4238640$	$(167y_{n+1} - y_{n+3}, 8y_{n+3} - 988y_{n+1})$
15	$35X^2 - Y^2 = 29435$	$(167y_{n+2} - 14y_{n+3}, 83y_{n+3} - 988y_{n+2})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabola	(X, Y)
1	$29X - 70Y^2 = 841$	$(8x_{2n+3} - 83x_{2n+2}, 14x_{n+1} - x_{n+2})$
2	$174X - 35Y^2 = 60552$	$(8x_{2n+4} - 988x_{2n+2}, 167x_{n+1} - x_{n+3})$
3	$29X - 70Y^2 = 841$	$(8y_{2n+2} - 35x_{2n+2}, 8x_{n+1} - y_{n+1})$
4	$87X - 35Y^2 = 15138$	$(8y_{2n+3} - 490x_{2n+2}, 83x_{n+1} - y_{n+2})$
5	$2059X - 35Y^2 = 4239481$	$(8y_{2n+4} - 5845x_{2n+2}, 988x_{n+1} - y_{n+3})$
6	$29X - 70Y^2 = 841$	$(83x_{2n+4} - 988x_{2n+3}, 167x_{n+2} - 14x_{n+3})$
7	$87X - 35Y^2 = 15138$	$(83y_{2n+2} - 35x_{2n+3}, 8x_{n+2} - 14y_{n+1})$
8	$29X - 70Y^2 = 841$	$(83y_{2n+3} - 490x_{2n+3}, 83x_{n+2} - 14y_{n+2})$
9	$87X - 35Y^2 = 15138$	$(83y_{2n+4} - 5845x_{2n+3}, 988x_{n+2} - 14y_{n+3})$
10	$2059X - 70Y^2 = 4239481$	$(988y_{2n+2} - 35x_{2n+4}, 8x_{n+3} - 167y_{n+1})$
11	$87X - 35Y^2 = 15138$	$(988y_{2n+3} - 490x_{2n+4}, 83x_{n+3} - 167y_{n+2})$
12	$29X - 70Y^2 = 841$	$(988y_{2n+4} - 5845x_{2n+4}, 988x_{n+3} - 167y_{n+3})$
13	$1015X - 2Y^2 = 29435$	$(14y_{2n+2} - y_{2n+3}, 83y_{n+1} - 8y_{n+2})$
14	$6090X - Y^2 = 2119320$	$(167y_{2n+2} - y_{2n+4}, 988y_{n+1} - 8y_{n+3})$
15	$1015X - 2Y^2 = 29435$	$(167y_{2n+3} - 14y_{2n+4}, 988y_{n+2} - 83y_{n+3})$

3. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola, represented by positive pell equation is given by $y^2 = 35x^2 + 29$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Positive Pell equations and determine their integer solutions along with suitable properties.

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